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A viscous-convective instability in laminar Keplerian thin discs. II. Anelastic approximation.

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ABSTRACT
Using the anelastic approximation of linearised hydrodynamic equations, we investigate the development of axially symmetric small perturbations in thin Keplerian discs. The sixth-order dispersion equation is derived and numerically solved for different values of relevant physical parameters (viscosity, heat conductivity, disc semi-thickness and vertical structure). The analysis reveals the appearance of two overstable modes which split out from the classical Rayleigh inertial modes in a wide range of the parameters in both ionized and neutral gases. These modes have a viscous-convective nature and can serve as a seed for turbulence in astrophysical discs even in the absence of magnetic fields.

Key words: hydrodynamics, instabilities, accretion discs

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1 INTRODUCTION

In an attempt to understand hydrodynamic instabilities which can potentially initiate turbulence in accretion discs, in Shakura & Postnov (2015) (Paper I) we have performed a local WKB-analysis of axially symmetric perturbations in thin accretion discs. It was found that under a special choice of wave vector direction of the perturbations, almost (but not completely) aligned with the disc symmetry plane, the presence of a microphysical viscosity, parametrized in terms of the mean-free path length to the disc scale ratio and taken into account in the dissipation function in the right-hand side of the energy equation, leads to the appearance of an overstable oscillating behaviour of one of two classical Rayleigh inertial modes. The instability was found in a wide range of perturbation wavelengths (expressed through the dimensional wavelength vector kr, where r is the disc radial scale), around $kr \sim 30-100$, in both fully ionized gases and neutral gases. The microphysical heat conductivity was taken into account through the dimensionless Prandtl number, Pr, which ranges from 0.052 for fully ionized plasma to 2/3 for neutral gases. We have found that the instability increment, reaching ~ 0.1 local Keplerian values, diminishes with decreasing the Prandtl number (e.g. due to the presence of a photon heat conductivity) and with increasing background vertical entropy gradient (expressed in terms of the Brunt-Väisälä frequency). Such a behaviour of the instability is in agreement with physically intuitive dumping effect of the heat conductivity and entropy gradients on the development of small radial perturbations propagating under a small angle to the disc plane.

To make the physics as simple as possible, in Paper I we have used the Boussinesq approximation of the hydrodynamic equations, which assumes the incompressibility of the fluid in the continuity equation and neglects the Euler pressure variations in the energy equation. We argued that the incompressibility approximation is justified for radial perturbations with $kr \gg 1$, which may suggest that the discovered viscous instability of the inertial Rayleigh modes is real and not the result of the approximations used.

In this paper we continue studying the viscous-convective instability in thin shear laminar flows found in our paper Shakura & Postnov (2015). Here we treat the problem in the anelastic approximation and take into account the vertical boundary conditions in the thin Keplerian discs. The anelastic approximation is the next approximation to the full system of hydrodynamic equations, but in which the term $\partial\rho/\partial t = 0$, i.e. the continuity equation takes the form $div(\rho\boldsymbol{u}) = 0$, which allows one to filter out sound waves.

When considering sound-proof stratified flows, the use of the anelastic approximation is known to have some subtleties (see, for example, the recent analysis by Vasil et al. (2013) and references therein). Special attention should be given to the energy equation, since the standard anelastic set of equations operates with adiabatic perturbations Ogura & Phillips (1962). Our analysis, in contrast, is heavily based on the viscous energy generation in the sheared flows, therefore the rigorous proof of the applicability of the anelastic approximation in this case is to be found. As a justification of this treatment we heuristically use the criterion that the linearised equations should not give rise to spurious modes with unphysical behaviour (e.g., unstable modes in the steady-state solid-body rotation case).

Although our analysis is applicable for any sheared axially symmetric flow, we will be mostly concerned with thin Keplerian discs which have a wide range of phenomenological applications. This means that in the continuity equation of importance becomes the term $\sim 1/\rho_0(\partial\rho_0/\partial z)$, which can be quite significant in thin discs and which we have neglected in the Boussinesq approximation. The vertical boundary conditions in thin discs are taken into account by solving the Sturm-Liouville problem for the z-part of perturbations.

The main result of the paper is the dispersion equation Eq. (48), which is a sixth-order algebraic equation for small perturbations in the form $f(z)\exp(\mathrm{i}\omega t - k_r r)$). The solution of this equation signals the appearance of an overstable behaviour for two modes with the same negative imaginary part and real parts with equal absolute values but different signs which split from two classical inertial Rayleigh modes, in a wide range of $k_r r \gg 1$.

The structure of the paper is as follows. In Section 2 we write down the basic equations. In Section 3 we linearise the full system of equation in the anelastic approximation. We proceed with the derivation of the dispersion equation in Section 4, following by its numerical analysis in Section 5. Section 6 summarizes our findings.

2 BASIC EQUATIONS

The system of hydrodynamic equations reads:

(i) mass conservation equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0\,, \tag{1}$$

In cylindrical coordinates for axially symmetric flows:

$$\nabla\cdot(\rho\boldsymbol{u}) = \frac{1}{r}\frac{\partial(\rho r u_r)}{\partial r} + \frac{\partial(\rho u_z)}{\partial z} \tag{2}$$

(ii) Navier-Stokes equation including gravity force

$$\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\nabla)\cdot\boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla\phi_g + \boldsymbol{\mathcal{N}}\,. \tag{3}$$

Here $\phi_g = -GM/r$ is the Newtonian gravitational potential of the central body with mass M, $\boldsymbol{\mathcal{N}}$ is the viscous force. In cylindrical coordinates for axially symmetric flows:

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\phi^2}{r} = -\frac{\partial\phi_g}{\partial r} - \frac{1}{\rho}\frac{\partial p}{\partial r} + \mathcal{N}_r\,, \tag{4}$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + u_z\frac{\partial u_\phi}{\partial z} + \frac{u_r u_\phi}{r} = \mathcal{N}_\phi\,, \tag{5}$$

$$\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}=-\frac{\partial\phi_g}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}+\mathcal{N}_z\,. \tag{6}$$

The linearised viscous force components are specified in Appendix A of Paper I.

(iii) energy equation

$$\frac{\rho\mathcal{R}T}{\mu}\left[\frac{\partial s}{\partial t}+(\boldsymbol{u}\nabla)\cdot s\right]=Q_{\rm visc}-\nabla\cdot\boldsymbol{F}\,. \tag{7}$$

where s is the specific entropy per particle, $Q_{\rm visc}$ is the viscous dissipation rate per unit volume, $\mathcal{R}$ is the universal gas constant, μ is the molecular weight, T is the temperature, and terms on the right stand for the viscous energy production and the heat conductivity energy flux $\boldsymbol{F}$, respectively. The energy flux due to the heat conductivity is

$$\nabla\cdot\boldsymbol{F}=\nabla(-\kappa\nabla T)=-\kappa\Delta T-\nabla\kappa\cdot\nabla T\,. \tag{8}$$

Note that both electrons and photons, and at low temperatures neutral atoms, can contribute to the heat conductivity (see Section 5 below).

(iv) equation of state

The equation of state for a perfect gas is convenient to write in the form:

$$p=Ke^{s/c_V}\rho^\gamma\,, \tag{9}$$

where K is a constant, $c_V=1/(\gamma-1)$ is the specific volume heat capacity and $\gamma=c_p/c_V$ is the adiabatic index (5/3 for the monoatomic gas). We will also use the equation of state in the form

$$p=\frac{\rho\mathcal{R}T}{\mu}\,, \tag{10}$$

where μ is the molecular weight.

3 LINEARISED EQUATIONS IN ANELASTIC APPROXIMATION

The perturbed hydrodynamic variables can be written in the form $x=x_0+x_1$, where x_0 stand for the unperturbed quantities and $x_1=(\rho_1,p_1,u_{r,1},u_{z,1},u_{\phi,1})$ are small perturbations. In contrast to Paper I in which we considered the local WKB approximation, i.e. small perturbations of density, pressure and velocity in the form $x_1(t,z,r)\propto\exp(\mathrm{i}\omega t-\mathrm{i}k_rr-\mathrm{i}k_zz)$, here we will take them in the form $\propto f(z)\exp(\mathrm{i}\omega t-\mathrm{i}k_rr)$ with the boundary conditions $f(z_0)=0$, $f(-z_0)=0$, where z_0 is the disc semi-thickness. We will consider thin discs with $z_0/r\sim u_s/u_{\phi,0}\ll 1$ (u_s is the sound velocity). Below we shall omit subscript 1 for small perturbations of the velocity, unless stated otherwise.

In this Section we will formultae the so-called anelastic approximation of hydrodynamic equations in which the sound wave perturbations are neglected by omitting the term $\partial\rho/\partial t$ in the continuity equation Ogura & Phillips (1962).

The linearised hydrodynamic equations are written as follows.

(i) **Continuity equation**

The anelastic approximation for gas velocity $\boldsymbol{u}$ is $\nabla\cdot\rho_0\boldsymbol{u}=0$:

$$\frac{\partial u_z}{\partial z}-\mathrm{i}k_ru_r+\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z}u_z+\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial r}u_r=0\,. \tag{11}$$

(ii) **Dynamic equations**

The radial, azimuthal and vertical components of the Navier-Stokes momentum equation are, respectively:

$$\mathrm{i}\omega u_r-2\Omega u_\phi=\mathrm{i}k_r\frac{p_1}{\rho_0}+\frac{\rho_1}{\rho_0^2}\frac{\partial p_0}{\partial r}-\nu k_r^2u_r+\nu\frac{\partial^2u_r}{\partial z^2}\,, \tag{12}$$

$$\mathrm{i}\omega u_\phi+\frac{\varkappa^2}{2\Omega}u_r=-\nu k_r^2u_\phi+\nu\frac{\partial^2u_\phi}{\partial z^2}\,, \tag{13}$$

$$\mathrm{i}\omega u_z=-\frac{1}{\rho}\frac{\partial p_1}{\partial z}+\frac{\rho_1}{\rho_0^2}\frac{\partial p_0}{\partial z}-\nu k_r^2u_z+\nu\frac{\partial^2u_z}{\partial z^2} \tag{14}$$

Here

$$\varkappa^2=4\Omega^2+r\frac{d\Omega^2}{dr}\equiv\frac{1}{r^3}\frac{d\Omega^2r^4}{dr} \tag{15}$$

is the epicyclic frequency. For the power-law rotation $\Omega^2\sim r^{-q}$ the epicyclic frequency is simply $\varkappa^2/\Omega^2=4-q$.

In deriving these equations we have set to unity the correction factors $[R],[\Phi],[Z],[E]$ introduced in Paper I and which take into account the dependence of the viscosity coefficient on temperature $\eta\sim T^{\alpha_{visc}}$ ($\alpha_{visc}=5/2$ for fully ionized gas and $\alpha_{visc}=1/2$ for neutral gas) in the perturbed viscous force component $\mathcal{N}_r,\mathcal{N}_\phi,\mathcal{N}_z$, respectively (see Appendix A and Eq. (42) in Paper I), because the deviations of these

coefficients from unity have a very insignificant effect on the results. Below we shall also neglect the second derivatives with respect to z of the perturbed velocity components in the dynamical equations (12)- (14). This can be justified if $\nu k_r^2 u_{r,\phi,z} > \nu(\partial^2 u_{r,\phi,z}/\partial z^2)$. To within a numerical factor, this inequality can be recast to the form $(k_r r)^2 (z_0/r)^2 > 1$. We shall see below that for $k_r r \gtrsim 100$ where the maximum instability increments occur and for thin discs with $z_0/r \sim 0.02$ (see Eq. (49)) this is indeed the case.

(iii) **Pressure and entropy perturbations**

In the general case by varying the equation of state Eq. (9) we obtain for entropy perturbations:

$$\frac{p_1}{p_0} = \frac{s_1}{c_V} + \gamma \frac{\rho_1}{\rho_0} . \tag{16}$$

On the other hand, from the equation of state for ideal gas in the form $p = \rho \mathcal{R} T/\mu$, we find for small temperature perturbations we have:

$$\frac{p_1}{p_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0} . \tag{17}$$

(iv) **Energy equation**

The linearised viscous dissipation function is

$$Q_{\rm visc} = \nu \rho r \frac{d\Omega}{dr} \left[r \frac{d\Omega}{dr} - 2{\rm i} k_r u_\phi - 2\frac{u_\phi}{r} \right] + \text{quadratic terms} . \tag{18}$$

Here $\Omega = u_{\phi,0}/r$ is the angular (Keplerian) velocity of the unperturbed flow. The linearised energy equation takes the form

$$\frac{\rho_0 \mathcal{R} T_0}{\mu} \left({\rm i}\omega s_1 + u_z \frac{\partial s_0}{\partial z} + u_r \frac{\partial s_0}{\partial r} \right) = -2{\rm i} k_r \nu \rho_0 r \frac{d\Omega}{dr} u_\phi - \kappa k_r^2 T_0 \frac{T_1}{T_0} , \tag{19}$$

To take into account the heat conductivity effects, it is convenient to introduce the dimensionless Prandtl number:

$$\Pr \equiv \frac{\nu \rho_0 C_p}{\kappa} = \frac{\nu \rho_0 (\mathcal{R}/\mu) c_p}{\kappa} = \frac{\nu \rho_0 (\mathcal{R}/\mu)}{\kappa} \frac{\gamma}{\gamma - 1} . \tag{20}$$

The Prandtl number defined by Eq. (20) for fully ionized hydrogen gas ($\gamma = 5/3$), where the heat conduction is determined by light electrons, is quite low (see Spitzer (1962)):

$$\Pr_{\rm e} \approx \frac{0.406}{20 \cdot 0.4 \cdot 0.225 \cdot (2/\pi)^{\frac{3}{2}}} \left(\frac{m_e}{m_p} \right)^{1/2} \left(\frac{5}{2} \right) \approx 0.052 . \tag{21}$$

Note that the presence of magnetic field in plasma decreases both electron heat conductivity and viscosity. In this case both the viscosity and heat conductivity are determined by ions that have larger Larmor radius than electrons, and the Prandtl number even in the case of fully ionized gas becomes Spitzer (1962)

$$\Pr_{\rm i} = \frac{3}{20} c_p \tag{22}$$

which is 3/8 for $\gamma = 5/3$.

In the case of cold neutral gas the Prandtl number is $\Pr_n$=2/3 according to simplified kinetic theory (Hirschfelder, Curtiss & Bird 1954), and the heat conductivity coefficient depends on temperature as $\kappa \sim T^{1/2}$ (Spitzer 1962).

After eliminating the temperature variations in the energy equation using Eq. (16) and Eq. (17), we find

$$\frac{\rho_1}{\rho_0} \left({\rm i}\omega + \frac{\nu k_r^2}{\Pr} \right) - \frac{1}{c_p} \left(u_z \frac{\partial s_0}{\partial z} + u_r \frac{\partial s_0}{\partial r} \right) = \frac{2{\rm i} k_r \nu r (d\Omega/dr)}{c_p \mathcal{R} T_0/\mu} u_\phi + \frac{p_1}{p_0} \left({\rm i}\frac{\omega}{\gamma} + \frac{\nu k_r^2}{\Pr} \right) \tag{23}$$

Here $c_p = \gamma c_V = \gamma/(\gamma - 1)$ is the specific heat capacity (per particle) at constant pressure.

We will neglect very slow variations of the unperturbed pressure, density and entropy along the radial coordinate, i.e. set $\partial/\partial r = 0$ in the continuity equations (11), dynamic equations (12)- (14) and energy equation (23). Below we shall also denote the partial derivative with respect to z by prime. Thus we are left with the following system of five linearised hydrodynamic equations in the anelastic approximation for five variable $u_r, u_z, u_\phi, \rho_1/\rho_0, p_1/p_0$:

$$u_z' - {\rm i} k_r u_r + \frac{\rho_0'}{\rho_0} u_z = 0 , \tag{24}$$

$$({\rm i}\omega + \nu k_r^2) u_r - 2\Omega u_\phi = {\rm i} k_r \frac{p_1}{\rho_0} , \tag{25}$$

$$({\rm i}\omega + \nu k_r^2) u_\phi + \frac{\varkappa^2}{2\Omega} u_r = 0 , \tag{26}$$

$$({\rm i}\omega + \nu k_r^2) u_z = -\frac{p_1'}{\rho_0} + \frac{\rho_1}{\rho_0} \frac{p_0'}{\rho_0} , \tag{27}$$

$$\frac{\rho_1}{\rho_0} \left({\rm i}\omega + \frac{\nu k_r^2}{\Pr} \right) = \frac{2{\rm i} k_r \nu r (d\Omega/dr)}{c_p \mathcal{R} T_0/\mu} u_\phi + \frac{1}{c_p} s_0' u_z . \tag{28}$$

In the right-hand side of energy equation (28) we have omitted the term due to pressure perturbations $\propto p_1/p_0$ because otherwise it will give rise to spurious unstable modes in the case of steady solid-body rotation with $\Omega = const$.

4 DERIVATION OF THE DISPERSION EQUATION

We start with substituting ρ_1/ρ_0 from Eq. (28) into Eq. (27). Here in the right-hand side of the resulting equation two coefficients depending on the z-coordinate arise:

$$\Phi_0 \equiv \frac{p_0'}{p_0}\nu \tag{29}$$

and the Brunt-Väisälä frequency:

$$-N_z^2 \equiv \frac{p_0'}{\rho_0}\frac{s_0'}{c_p} = \frac{p_0'}{\rho_0}\frac{\partial}{\partial z}\left(\frac{p_0^{1/\gamma}}{\rho_0}\right). \tag{30}$$

After differentiating the resulting equation for u_z with respect to z and eliminating u_z', u_r and u_ϕ using (24)- (27), we arrive at the following second-order linear differential equation for density perturbations

$$p_1'' + Ap_1' + Bp_1 = 0 \tag{31}$$

with coefficients:

$$A = \frac{(-N_z^2)'}{(\mathrm{i}\omega + \frac{\nu k_r^2}{\mathrm{Pr}})(\mathrm{i}\omega + \nu k_r^2) - (-N_z^2)} - \Phi_0 \frac{\varkappa^2}{(\mathrm{i}\omega + \nu k_r^2)^2} \frac{(d\ln\Omega/d\ln r)}{c_p(\mathrm{i}\omega + \frac{\nu k_r^2}{\mathrm{Pr}})} \frac{k_r^2}{1 + \frac{\varkappa^2}{(\mathrm{i}\omega + \nu k_r^2)^2}} \tag{32}$$

$$B = \frac{-k_r^2}{1 + \frac{\varkappa^2}{(\mathrm{i}\omega + \nu k_r^2)^2}} \left[1 - \frac{(-N_z^2)}{(\mathrm{i}\omega + \nu k_r^2)(\mathrm{i}\omega + \frac{\nu k_r^2}{\mathrm{Pr}})} + \Phi_0' \frac{\varkappa^2}{(\mathrm{i}\omega + \nu k_r^2)^2} \frac{(d\ln\Omega/d\ln r)}{c_p(\mathrm{i}\omega + \frac{\nu k_r^2}{\mathrm{Pr}})} \left(1 + \left(\frac{\Phi_0}{\Phi_0'}\right) \frac{(-N_z^2)'}{(\mathrm{i}\omega + \frac{\nu k_r^2}{\mathrm{Pr}})(\mathrm{i}\omega + \nu k_r^2) - (-N_z^2)}\right)\right] \tag{33}$$

Let us introduce a new variable

$$p_1 = Ye^{-\frac{1}{2}Az} \tag{34}$$

to eliminate the first derivative term in Eq. (31):

$$Y'' + (B - \frac{1}{4}A^2)Y = 0\,. \tag{35}$$

This equation should be supplemented with two boundary conditions:

$$Y_1(z_0) = 0, \quad Y_1(0) = 0 \tag{36}$$

or

$$Y_2(z_0) = 0, \quad Y_2'(0) = 0 \tag{37}$$

for two linearly independent solutions. Using the form of the coefficient A (32) it is easy to check that these boundary conditions exactly correspond to the physically motivated boundary conditions for pressure variations: $p_1(z_0) = 0$, $p_1(0) = 0$ and $p_1(z_0) = 0$, $p_1'(0) = 0$, respectively.

This Sturm-Liouville problem (35)- (37) can be easily solved if coefficients A and B are independent of z. Therefore, it is necessary to average these coefficients over some assumed background disc vertical structure. As the background solution, we will use the polytrope discs with (see Ketsaris & Shakura (1998)):

$$\rho_0(z) = \rho_c\left(1 - \left(\frac{z}{z_0}\right)^2\right)^n, \quad p_0(z) = p_c\left(1 - \left(\frac{z}{z_0}\right)^2\right)^{(n+1)}, \quad T_0(z) = T_c\left(1 - \left(\frac{z}{z_0}\right)^2\right). \tag{38}$$

Here n is the polytrope index, ρ_c, p_c and T_c are the values of density, pressure and temperature in the disc symmetry plane, respectively. Therefore, the density-averaged values of quantities Φ_0, Φ_0' and $(-N_z^2)$ should be determined as

$$\langle(...)\rangle \equiv \frac{\int_0^{z_0}(...)\rho_0(z)dz}{\int_0^{z_0}\rho_0(z)dz} \tag{39}$$

where z_0 is the semi-thickness of the disc. Thus we obtain:

$$\Phi_0 = \langle\Phi_0\rangle\frac{\Omega a r^2}{z_0}\,, \tag{40}$$

where, as in Paper I, we have introduced the dimensionless viscosity parameter a through the free-path length of particles l/r and the ratio of the sound velocity to the unperturbed angular (Keplerian) velocity u_s/u_ϕ

$$\frac{\nu k_r^2}{\Omega} = a(k_r r)^2\,, \quad a \equiv \left(\frac{u_s}{u_\phi}\right)\left(\frac{l}{r}\right)\,. \tag{41}$$

Note that the maximum possible mean-free path in thin discs in the frame of hydrodynamic treatment should be less than the disc thickness z_0, i.e. $l/r = (l/z_0)(z_0/r) \simeq (l/z_0)(u_s/u_\phi) < (u_s/u_\phi)$. The derivative of Φ_0 can be written as

$$\Phi_0' = \langle\Phi_0'\rangle\frac{\Omega a r^2}{z_0^2}\,, \tag{42}$$

The corresponding dimensionless mean values are

$$\langle\Phi_0\rangle = -\frac{n+1}{\alpha_{visc}\langle\Sigma_0\rangle} \tag{43}$$

$$\langle\Phi_0'\rangle = \frac{1}{\langle\Sigma_0\rangle}\left(2(n+1)(n+2)B(\frac{3}{2},\alpha_{visc}-1) + \frac{n(n+1)}{\alpha_{visc}-1} - 2(n+1)nB(\frac{3}{2},n-1)\right)\,. \tag{44}$$

Similarly, for the mean Brunt-Väisälä frequency we find:

$$\langle -N_z^2\rangle = \frac{2\Omega^2}{\langle\Sigma_0\rangle}\left(\frac{n+1}{\gamma} - n\right)B(\frac{3}{2},n)\,. \tag{45}$$

In the above formulas the dimensionless surface density of the disc is

$$\langle\Sigma_0\rangle \equiv \frac{\int_0^{z_0}\rho_0 dz}{\rho_c z_0} = 2^{2n}B(n+1,n+1)\,, \tag{46}$$

$B(x,y)$ is the beta-function. When deriving these values, we have used the property that the microscopic dynamic viscosity is a function of temperature only $\nu_0\rho_0 \sim T^{\alpha_{visc}}$, with $\alpha_{visc} = 5/2$ for fully ionized gas. For neutral gas $\alpha_{visc} = 1/2$ and the averaging should be performed with weight ρ_0^2 (see below in Section 5.2).

The solution of the Sturm-Liouville problem Eq. (35) with boundary conditions (36) and (37) results in eigenfunctions $\sin(\lambda_s z)$ and $\cos(\lambda_c z)$ and eigenvalues $\lambda_s = m\pi/z_0$ and $\lambda_c = (m-1/2)\pi/z_0$ ($m = 1,2,...$)), respectively. Substituting the eigenfunctions into Eq. (35) yields the sought for dispersion equation:

$$-\lambda_{s,c}^2 + B - \frac{1}{4}A^2 = 0\,. \tag{47}$$

The form of the coefficient A (32) immediately implies that the dispersion equation will be a tenth-order algebraic equation for ω with complex coefficients. It can be checked that for any value of $n > 3/2$ (i.e. for convectively stable disc structure), two spurious unstable modes arise in the case of steady-state solid-body rotation (with $q = 0$), which disappear if we omit the terms with $(-N_z^2)'$ (note that this term is not dangerous for adiabatic background structure where $(-N_z^2)$ and its derivative vanish). Similarly unstable modes would arise if we retain pressure perturbations in the right-hand side of the energy equation (23). Due to subtleties with the energy equation in the Boussinesq and anelastic approximations mentioned above and analysed in (Vasil et al. 2013), we conclude that the Brunt-Väisälä frequency should be kept constant when differentiating Eq. (27). This might indicate that the background entropy gradient s_0' should be omitted in the right-hand side of the energy equation (28) from the very beginning. Not so, since its inclusion leads to the physically correct result (stabilization of the modes) already in the Boussinesq approximation (see Paper I). Another possibility might be the keeping both Φ_0 and $(-N_z^2)$ constant when deriving equation (31) for pressure perturbations. Not so again, since the factor Φ_0 (29) stems from the fully leguitime linearised form of $z-$component of the Navier-Stokes equation (14). While, of course, rigorous proof of such a treatment is highly desirable, here we restrict ourselves to the qualitative arguments given above. Therefore, after crossing out terms with $(-N_z^2)'$ in Eq. (32) and Eq. (33), we will be left with a sixth-order algebraic equation for ω.

The found eigenfunctions for the variable Y means that the eigenfunctions for the pressure and velocity perturbations have the form $\exp(-Az/2)\cos(\lambda_s z)$ or $\exp(-Az/2)\sin(\lambda_s z)$, which is typical for perturbations in stratified atmospheres Vasil et al. (2013). We will find that the maximum increment is reached for the cos mode, so substituting coefficients A and B from Eq. (32) and Eq. (33) yields the following quantized dispersion equation:

$$\begin{aligned}-\left(\frac{(m-1/2)\pi}{z_0}\right)^2 - \frac{k_r^2}{\left(1+\frac{\varkappa^2}{(\mathrm{i}\omega+\nu k_r^2)^2}\right)}\left[1 - \frac{(-N_z^2)}{(\mathrm{i}\omega+\nu k_r^2)(\mathrm{i}\omega+\frac{\nu k_r^2}{\mathrm{Pr}})} + \Phi_0'\frac{\varkappa^2}{(\mathrm{i}\omega+\nu k_r^2)^2}\frac{(d\ln\Omega/d\ln r)}{c_p(\mathrm{i}\omega+\frac{\nu k_r^2}{\mathrm{Pr}})}\right]\\ -\frac{1}{4}\left[-\Phi_0\frac{\varkappa^2}{(\mathrm{i}\omega+\nu k_r^2)^2}\frac{(d\ln\Omega/d\ln r)}{c_p(\mathrm{i}\omega+\frac{\nu k_r^2}{\mathrm{Pr}})}\frac{k_r^2}{\left(1+\frac{\varkappa^2}{(\mathrm{i}\omega+\nu k_r^2)^2}\right)}\right]^2 = 0\,.\end{aligned} \tag{48}$$

This is a sixth-order algebraic equation with complex coefficients (cf. cubic dispersion equation (31) from Paper I derived in the Boussinesq limit using local WKB-analysis).

5 SOLUTION OF THE DISPERSION EQUATION

Let us analyse the solution of the dispersion equation in the anelastic approximation derived above for two fluids: the case of fully ionized gas with the Prandtl number $\mathrm{Pr}_e = 0.053$ and $\mathrm{Pr}_i = 3/8$ if the magnetic field is present, and the case of neutral gas with $\mathrm{Pr}_n = 2/3$. The last case should be treated separately, since for the dependence of the dynamical viscosity $\propto T^{1/2}$ the averaging over the vertical coordinate with the weight $\rho_0(z)$ is insufficient – near the surface layers of the disc the mean free-path length of particles is so large that leads to divergences in the term Φ_0'.

We will consider the laminar shear flow with the velocity profile $\Omega^2 \propto r^{-q}$ so that the shear coefficient is $d\ln\Omega/d\ln r = -q/2$. All relevant frequencies will be normalized to the local Keplerian value Ω and denoted with tilde. In the numerics the adiabatic index of the gas is set to $\gamma = 5/3$.

It is convenient to write down the dimensionless dispersion equation for the dimensionless mode frequency $\tilde{\omega}$ as a function of the dimensionless variable $(k_r r)$ with dimensionless viscosity parameter a and the disc thickness

$$\frac{z_0}{r} = \sqrt{\frac{\Pi_1}{\gamma}}\left(\frac{u_s}{u_\phi}\right). \tag{49}$$

Here the dimensionless factor Π_1 takes into account the vertical disc structure, and in the case of the polytrope accretion discs $\Pi_1 = 2(n+1)$ (Ketsaris & Shakura 1998). The values of the sound velocity u_s then should be taken in the disc symmetry plane.

The dispersion equation (48) in the dimensionless form reads:

$$\left(\frac{(m-1/2)\pi}{z_0/r}\right)^2 + \frac{(k_r r)^2}{\left(1+\frac{\tilde{\varkappa}^2}{(\mathrm{i}\tilde{\omega}+a(k_r r)^2)^2}\right)}\left[1 - \frac{\langle -\tilde{N}_z^2\rangle}{(\mathrm{i}\tilde{\omega}+a(k_r r)^2)(\mathrm{i}\tilde{\omega}+\frac{a(k_r r)^2}{\mathrm{Pr}})} + \frac{\langle\Phi_0'\rangle a}{(z_0/r)^2}\frac{\tilde{\varkappa}^2}{(\mathrm{i}\tilde{\omega}+a(k_r r)^2)^2}\frac{(-q/2)}{c_p(\mathrm{i}\tilde{\omega}+\frac{a(k_r r)^2}{\mathrm{Pr}})}\right]$$
$$+\frac{1}{4}\left[\frac{\langle\Phi_0\rangle a}{(z_0/r)}\frac{\tilde{\varkappa}^2}{(\mathrm{i}\tilde{\omega}+a(k_r r)^2)^2}\frac{(-q/2)a}{c_p(\mathrm{i}\tilde{\omega}+\frac{a(k_r r)^2}{\mathrm{Pr}})}\frac{(k_r r)^2}{\left(1+\frac{\tilde{\varkappa}^2}{(\mathrm{i}\tilde{\omega}+a(k_r r)^2)^2}\right)}\right]^2 = 0. \tag{50}$$

5.1 Fully ionized gas

The results of the solution of the dispersion equation (50) are shown in Fig. 1 for two background vertical structures of a thin Keplerian disc. In the left panel of Fig. 1 we present the solution for the polytrope disc structure with constant entropy described by the polytrope index $n = \frac{3}{2}$. Here the Brunt-Väsälä frequency N_z^2 vanishes. It is seen that the unstable anelastic mode (the one with the negative imaginary part in the bottom panel) arises at $k_r \sim 50-150$, where the approximation of the incompressibility $(\partial\rho/\partial t = 0)$ is justified. In fact, this is two modes with equal absolute values but different by sign real parts that demonstrate the overstability (marked with arrows in the upper left panel). This is different from the Boussinesq limit considered in Paper I, where one of the Rayleigh inertial modes became viscously overstable in the presence of viscosity. In the anelastic approximation, the Rayleigh inertial modes remain always stable, and the unstable modes are split out from the Rayleigh modes. In the right panels of Fig. 1 we present the solutions for the background disc structure with vertically increasing entropy (shown is the solution for $n = 2$), which is convectively stable $(N_z^2 > 0)$ in the absence of viscosity and shear. However, the presence of even small viscosity makes the Keplerian flow convectively unstable even in this case.

In Fig. 2 we explore the effect of different dimensionless parameters of the problem on the increment of the overstability. First, in the left panel of Fig. 2 we study the effect of changing the Prandtl number, which describes the dumping effect of thermal conductivity on small perturbations. For fully ionized gas, the Prandtl number is maximal when the small (but still dynamically unimportant) magnetic field is present, and both the viscosity and heat conductivity are mediated by ions which have larger Larmor radius than electrons (Pr_i=3/8, see above). We also show the results of the dumping effect of possible radiative conductivity parameterized in terms of the effective Prandtl number (Pr/2 and Pr/11, see Eq. (47) in Paper I). The smaller the Prandtl number, the smaller the instability increment, which is physically clear. Second, in the central panel of Fig. 2 we show the effect of changing the viscosity (parametrized in terms of the effective mean free-path length of particles, l/r). The larger the viscosity, the higher the instability increment. Finally, in the left panel of Fig. 2 we demonstrate the effect of the disc semi-thickness (parametrized by the central sound speed to the angular velocity ratio, u_s/u_ϕ). At a given mean-free path of particles, the thinner the disc, the higher the increment.

5.2 Neutral gas

For fully neutral gas with the Prandtl number $\mathrm{Pr}_n = 2/3$ we should first make new averaging of quantities Φ_0, Φ_o' and $(-N_z^2)$ with weight $\rho_0^2(z)$ to avoid divergence of the value Φ_0' near the disc surface due to very large mean-free path length of particles in the polytropic discs. We find:

$$\langle\langle\Phi_0\rangle\rangle = -\frac{n+1}{(\alpha_{visc}+n)\langle\langle\Sigma_0\rangle\rangle} \tag{51}$$

$$\langle\langle\Phi_0'\rangle\rangle = \frac{1}{\langle\langle\Sigma_0\rangle\rangle}\left(2(n+1)(n+2)B(\frac{3}{2},\alpha_{visc}-1+n/2) + \frac{n(n+1)}{\alpha_{visc}-1+n} - 2(n+1)nB(\frac{3}{2},n-1+n/2)\right) \tag{52}$$

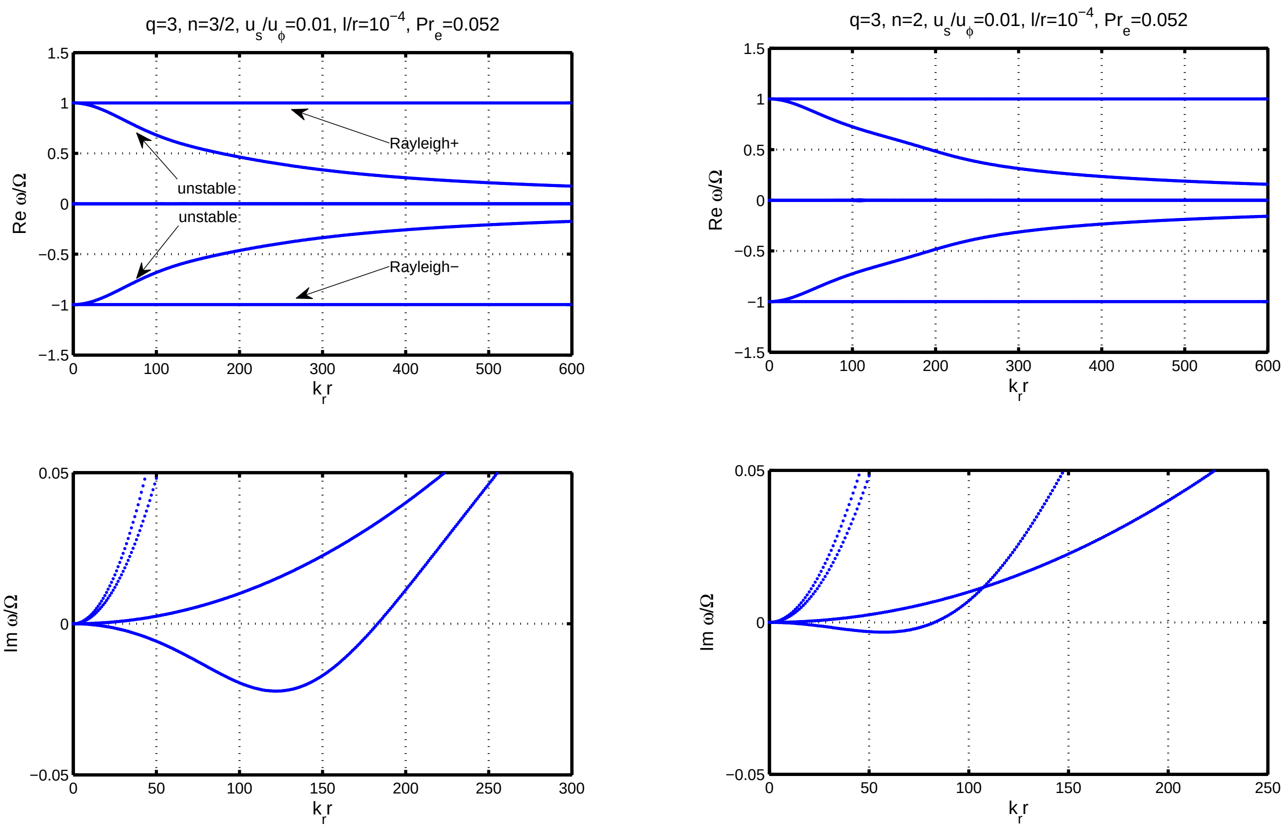

Figure 1. Left: Real and imaginary parts of anelastic modes in fully ionized gas with electron heat conductivity (Pr_e=0.052) and viscosity parameters $u_s/u_\phi = 0.01, l/r = 10^{-4}$ for the adiabatic density distribution ($n = \frac{3}{2}$). Right: The same for vertically increasing entropy distribution with $n = 2$.

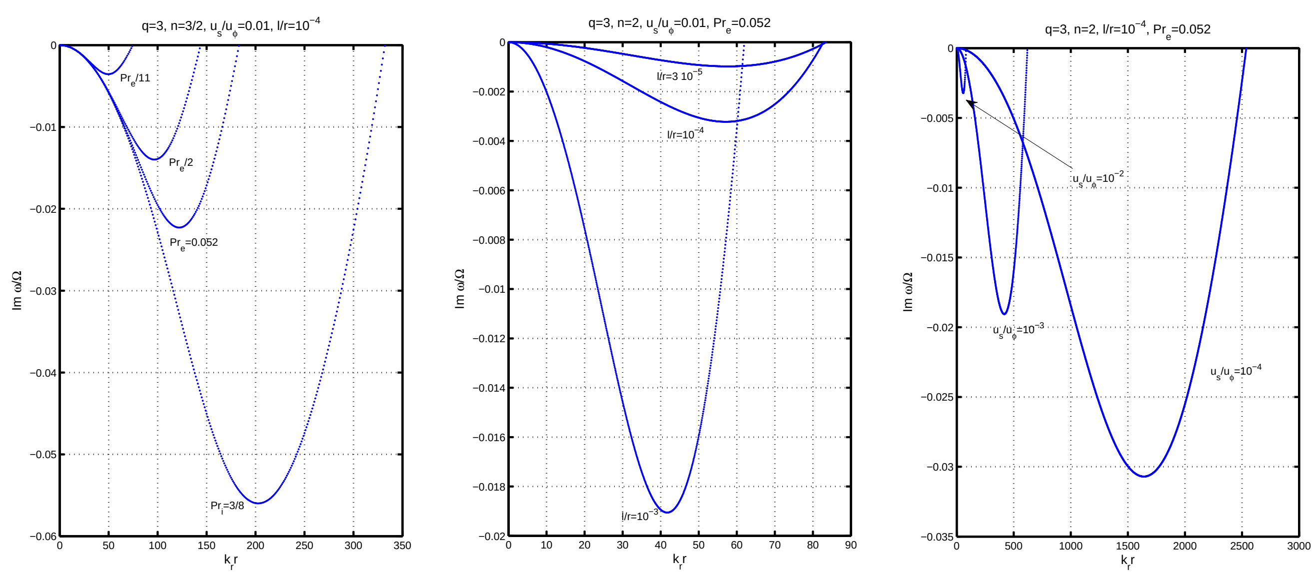

Figure 2. Left: Imaginary part of the viscously unstable anelastic mode in fully ionized gas with different Prandtl numbers. Viscosity parameters $u_s/u_\phi = 0.01, l/r = 10^{-4}$ for vertically increasing entropy distribution with $n = 2$. Middle: The same for different mean free-path length of particles l/r with fixed disc thickness parameter $u_s/u_\phi = 0.01$. Right: The same with different disc thickness parameter u_s/u_ϕ for fixed mean free-path length of particles $l/r = 10^{-4}$.

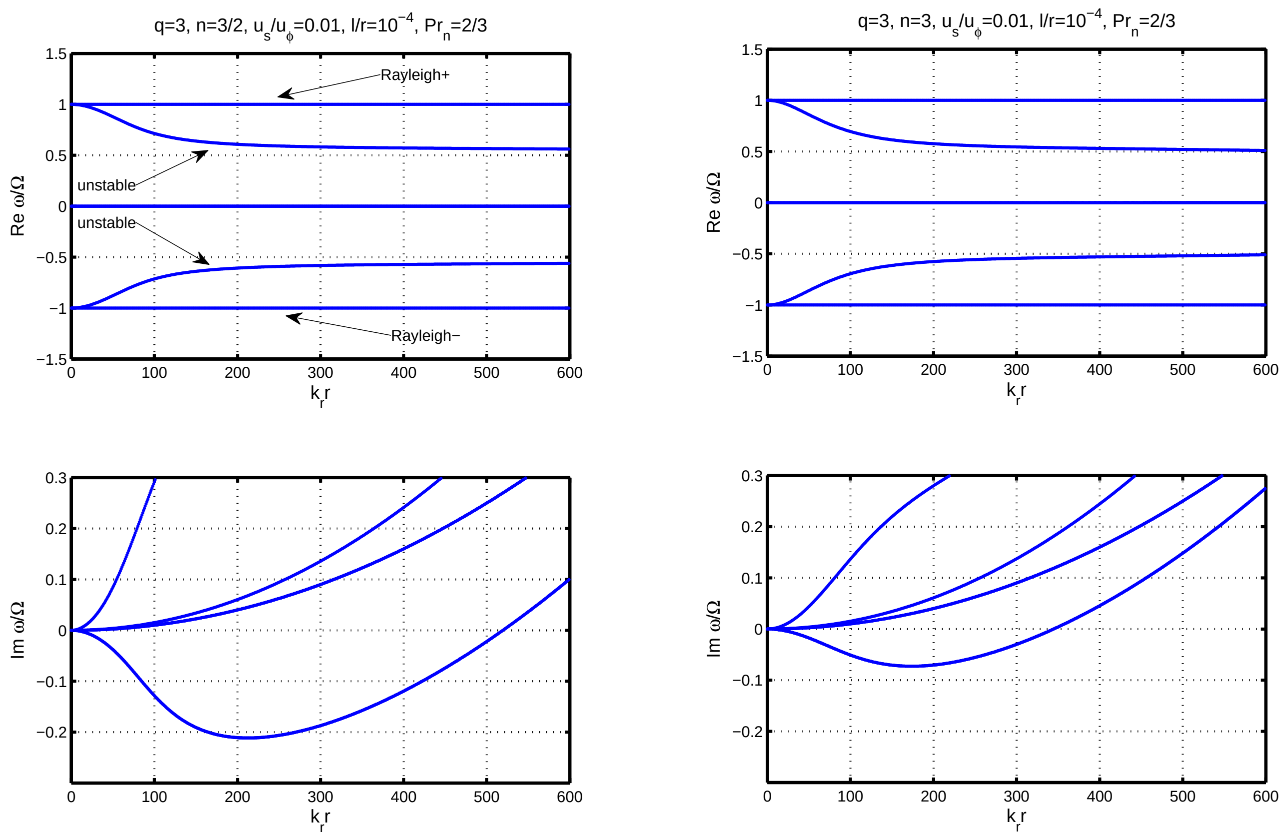

Figure 3. Left: Real and imaginary parts of anelastic modes in neutral gas for the adiabatic entropy distribution ($n = \frac{3}{2}$). Right: The same for vertically increasing entropy distribution with $n = 3$. Note that the real part of the solution is quite insensitive to the polytropic structure of the disc.

$$\langle\langle(-N_z^2)\rangle\rangle = \frac{2}{\langle\langle\Sigma_0\rangle\rangle}\left(\frac{n+1}{\gamma} - n\right) B(\frac{3}{2}, 2n)\,. \tag{53}$$

$$\langle\langle\Sigma_0\rangle\rangle = 2^{4n} B(2n+1, 2n+1)\,, \tag{54}$$

In Fig. 3 we show the real and imaginary parts of anelastic modes for the case of neutral gas with the Prandtl number $\mathrm{Pr}_n = 2/3$. The disc thickness is $z_0/r \sim u_s/u_\phi = 0.01$, the mean free path of ions is $l/r = 10^{-4}$. Since the Prandtl number is quite large, unlike the case of the ionized gas shown in Fig. 1, the instability is present even for the quite significant vertical background entropy gradient ($n = 3$, the right panel of this Figure).

6 DISCUSSION AND CONCLUSIONS

In the present paper we have extended the modal analysis of small axially symmetric perturbation in sheared Keplerian flows with microphysical viscosity and heat conductivity, which we started in Paper I. In contrast to Paper I, where the Boussinesq approximation was used, here we have formulated the problem in the anelastic approximation and taken into account vertical boundary conditions in thin discs. In this approximation we have obtained the second-order linear differential equation with respect to the z-coordinate for small pressure perturbations, Eq. (31), with coefficients (Eq. (32) and Eq. (33)) depending on the height above the disc plane z. We have assumed the background polytropic vertical disc structure and averaged the coefficients Eq. (32) and Eq. (33) with weight $\rho_0(z)$ over the vertical disc height. This allowed us to solve the Sturm-Liouville problem to obtain the discrete spectrum of eigenvalues and eigenfunctions ($\cos(\lambda_c z)$ or $\sin(\lambda_s z)$). Substitution of these functions into Eq. (31) resulted in the dispersion equation for normal modes of axially symmetric perturbations along the radial coordinate, Eq. (48). This turned out to be an algebraic sixth-order equation, solution of which for a wide range of the viscosity parameters in thin Keplerian discs are presented in Fig. 1-3. Note that in the Boussinesq limit the dispersion equation had only the third order. We have found that in a wide range of wavenumbers $k_r r \gg 1$ two unstable modes are split from the classical inertial Rayleigh modes (in the Boussinesq limit one of the Rayleigh modes displayed the overstability in the presence of viscosity). Qualitatively, the results of the present paper are in agreement with findings of Paper I. However, in contrast to Paper I, where the Boussinesq approximation was used, the results are quantitatively different.

The description of the hydrodynamic flows in the anelastic approximation, although neglects the sound modes, is more precise and takes into account the important term in the continuity equation, $(1/\rho_0\partial\rho_0/\partial z)u_z$. This means that the analysed small perturbations are no more purely transversal, as in the Boussinesq limit. However, it is important to note that in both anelastic and Boussinesq approximations the pressure variations p_1/p_0 should be neglected in the energy equation, otherwise fictitious unstable solutions emerge in the case of the steady-state solid-body rotation. We have also found that the overstability appears in the cases where there is a non-zero vertical gradient of the quantity $(p_0'/p_0)\nu \sim -(\Omega^2 z/T_0)\nu$.

In both approximation we have found that the increment of overstable modes increases with viscosity and the background vertical pressure gradient, suggesting a convective nature of the overstability: the viscous heat generation in the sheared flow in the gravitational field around a central star makes the flow convectively unstable. It is tempting to suggest that this instability may be the seed for turbulence in Keplerian discs even in the absence of magnetic fields.

Note the many faces of the viscosity in the considered problem. The higher the viscosity in the right-hand side of energy equation (28), the stronger the viscous energy generation due to the shear leading to the buoyancy of the perturbed regions. On the other hand, in the dynamic equations (12)- (14) the viscosity damps the perturbations. In the right-hand side of these equations, we have neglected terms $\nu u_r'', \nu u_\phi'', \nu u_z''$ with second derivatives of perturbed velocities. We expect that their taking into account will somewhat decrease the increment of the viscous-convective instability and narrow the interval of unstable wavenumbers in Figs. 1 and 3. If we retain these second derivatives, we will obtain a much more complicated sixth-order differential equation for perturbations. This is a separate problem to be addressed elsewhere. Here we have restricted ourselves to solving only the second-order differential equation Eq. (31).

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REFERENCES

Hirschfelder J. O., Curtiss C. F., Bird R. B., 1954, Molecular Theory of Gases and Liquids. J. Wiley and Sons, New York
Ketsaris N. A., Shakura N. I., 1998, Astronomical and Astrophysical Transactions, 15, 193
Ogura Y., Phillips N. A., 1962, Journal of Atmospheric Sciences, 19, 173
Shakura N., Postnov K., 2015, MNRAS, 448, 3707
Spitzer L., 1962, Physics of Fully Ionized Gases. Interscience, New York
Vasil G. M., Lecoanet D., Brown B. P., Wood T. S., Zweibel E. G., 2013, ApJ, 773, 169